VIRGO	N T S 9 5 0 1 6
Henrich Heitmann 24.04.1995	**A plane wave simulation tool for arbitrary interferometer configurations**

A procedure is described which allows the calculation of the field at any point in a complex system of coupled cavities, including interferometer systems. The light at the system input may be modulated (frontal modulation). This makes it possible to calculate error signals for stabilization of the positions of the optical elements. Based on this procedure, a computer program has been written. Its applications include the calculation of resonance behaviour and Pound-Drever error signals for simple or composite cavities and interferometers. It served already for calculation of signal matrices for the VIRGO longitudinal locking.

Overview

For calculation of the longitudinal locking scheme of VIRGO, a computer program (CAVITY) has been developed, which allows the plane-wave, quasi-stationary treatment of arbitrary systems of coupled cavities. This program and an analytical one, developed by R. Flaminio, were used for the simulation of the simple frontal modulation and SSB modulation techniques [1]. Since the underlying mathematical description of the system and the program itself are completely general and may be useful for a wide range of problems involving interferometers and coupled cavities, the principle of operation is described in the following. A brief overview over the functions of CAVITY is also given.

Locking

Longitudinal locking of VIRGO means keeping the relative positions of the mirrors sufficiently stable to maintain all interference conditions (resonance and dark fringe) necessary for the operation of the interferometer. For a simple cavity there exists the Pound-Drever stabilization technique [2], which consists in phase modulating the light incident on a cavity; a slight mismatch between the laser frequency and the cavity resonant frequency leads to an amplitude modulation in the reflected light, which is observed with a photodiode. Demodulation of the diode current yields an error signal for correcting cavity length or laser frequency. This scheme can be extended to more complicated resonant systems like the VIRGO interferometer, consisting of 4 coupled cavities with a beam splitter. Obviously, one photodiode is no more sufficient for obtaining the informations on all mass positions. In order to know where to place photodiodes, and which will be their signals, it is necessary to have a numerical tool which permits calculation of the light field in a complicated system for each of the side band frequencies appearing in the case of modulation of the input field.

Algorithm

Thus, we want to know the field at a given point (point of interest) in a system of coupled cavities. In order to be able to treat arbitrary systems, we use a general approach: Each optical element (object) is represented by a matrix describing its transmission and reflection properties. Neighboring objects can be combined, thereby stepwise reducing the number of matrices necessary for description of the system. So the whole system can finally be reduced to two composite objects, each described by a 2x2 matrix, with the point of interest between them. For this simple case the field can easily be calculated as a function of the input field.

If the input field is modulated, this means that it contains also sideband frequencies as modulation products. What we want to know is the modulation of the diode current, i.e. the beat note between some of the frequencies, observed at the point of interest. The procedure outlined above must therefore be repeated for those of the frequencies which contribute to the output signal to be calculated, which is then obtained by summing up the resulting individual fields in the appropriate way.

Repeating the whole calculation with slightly changed system configurations gives the dependence of each signal on specific mass positions, which can be expressed in form of a matrix.

Principle of calculation

2-port objects

Definition

As it was said above, for the purpose of modeling the interferometer is decomposed into a series of objects, which can be described by a matrix. The simplest case is an object with just two sides where a beam can be input or output (2-port objects, 2Ps). One example is a mirror (at normal incidence), where beams can come from the left or from the right, and each beam can be transmitted or reflected; the transmitted beam from one side is then superposed with the reflected beam from the other side. So one naturally comes to a representation with a 2x2-matrix **P**, where the element P_{ij} indicates the coupling factor between the field E_j incident on port j (= left or right side), and the field A_i exiting (transmitted or reflected) from port i.

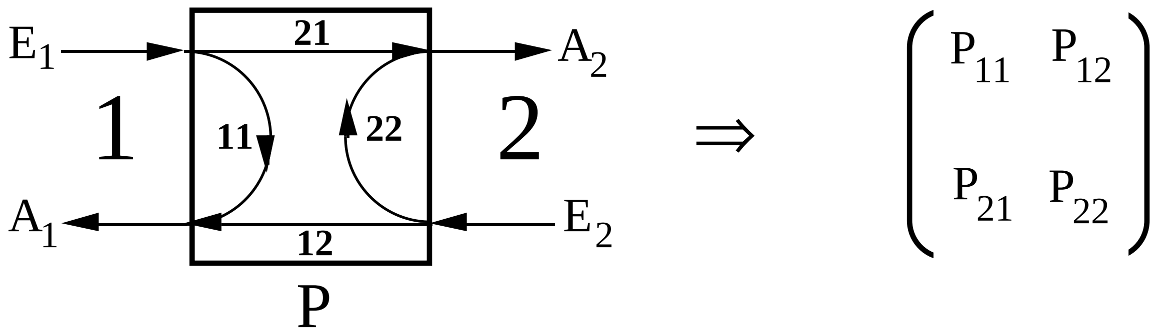

Fig. 1: Representation of an optical element as a 2-port object with the corresponding matrix

Calculating the outgoing fields A_1 and A_2 from the incident fields E_1 and E_2 can then be done by the multiplying the vector **E** with the matrix **P**:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \quad . \tag{1}$$

Examples for 2-port objects are:

mirror	**propagator**	**ideal optical isolator**
field transmission t field reflectivity r	distance d $k = 2\pi n/\lambda$	transmitting from left to right
$\begin{pmatrix} r & it \\ it & r \end{pmatrix}$	$\begin{pmatrix} 0 & e^{-ikd} \\ e^{ikd} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

According to these definitions, the unity matrix represents an ideal mirror with no transmission (no coupling between the two ports).

Concatenation

If one is just interested in the relation between the input and output of a chain of 2-port objects without caring about the internal fields, one can represent the system by a single 2x2 matrix. This matrix can be obtained by repeatedly performing the basic operation of combining adjacent 2x2 matrices, which will be described here.

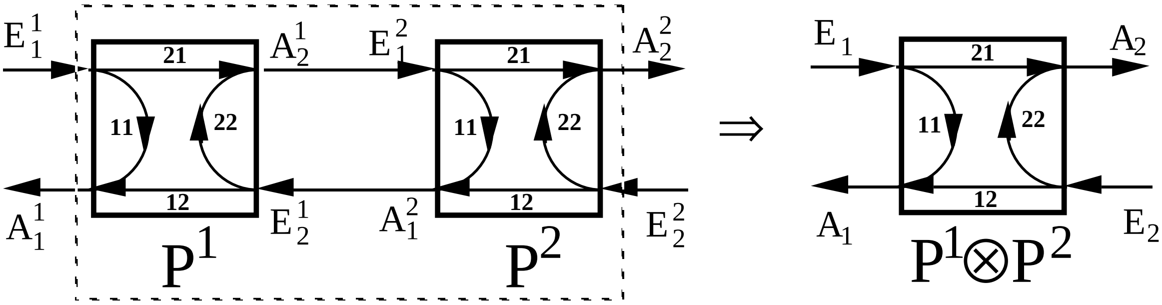

Fig. 2: Concatenation of two 2-port objects

In order to calculate the combined matrix $\mathbf{Q} = \mathbf{P^1} \otimes \mathbf{P^2}$, it is necessary to express the output vector **A** in terms of the input vector **E**:

$$\mathbf{A} = \begin{pmatrix} A_1^1 \\ A_2^2 \end{pmatrix} = \mathbf{Q}\,\mathbf{E} = \mathbf{Q} \begin{pmatrix} E_1^1 \\ E_2^2 \end{pmatrix} \quad . \tag{2}$$

The first element of the output vector is given by

$$A_1 = A_1^1 = E_1^1\, P_{11}^1 + E_2^1\, P_{12}^1 \quad ; \tag{3}$$

with

$$\left. \begin{aligned} E_2^1 &= A_1^2 = E_2^2\; P_{12}^2 + E_1^2\; P_{11}^2 \\ E_1^2 &= A_2^1 = E_1^1\; P_{21}^1 + E_2^1\; P_{22}^1 \end{aligned} \right\} E_2^1 = R\,(E_2^2\, P_{12}^2 + E_1^1\, P_{21}^1\, P_{11}^2\,) \tag{4}$$

one obtains

$$A_1 = E_1\,(P_{11}^1 + R\,P_{21}^1\,P_{11}^2\,P_{12}^1) + E_2\,R\,P_{12}^2\,P_{12}^1 \quad , \tag{5}$$

with

$$R = \frac{1}{1 - P_{11}^2 P_{22}^1} \quad . \tag{6}$$

A_2 is obtained likewise, and the final matrix is

$$\mathbf{Q} = \mathbf{P^1} \otimes \mathbf{P^2} = \begin{pmatrix} P_{11}^1 + R P_{21}^1 P_{11}^2 P_{12}^1 & R\,P_{12}^2 P_{12}^1 \\ R\,P_{21}^1 P_{21}^2 & P_{22}^2 + R P_{12}^2 P_{22}^1 P_{21}^2 \end{pmatrix} \quad . \tag{7}$$

Note that the matrix elements Q_{ij} can be determined following a set of intuitive rules:

1. Calculate the contribution for each possible path from port j to port i of the composite object by multiplying the matrix elements encountered on the way; so for Q_{11}: P_{11}^1 for reflection at P^1 and $P_{21}^1 P_{11}^2 P_{12}^1$ for P^1 (transm.), P^2 (refl.), P^1 (transm.).
2. If the path touches a loop (formed by two opposing reflections P_{11}^2 and P_{22}^1), multiply by the resonance factor $R = 1/(1-P_{11}^2 P_{22}^1)$; this takes account of the possibility to circle around the loop any number of times
3. Sum all paths.

Another representation of Eq. 7 is

$$\mathbf{Q} = \mathbf{P^1} \otimes \mathbf{P^2} = \begin{pmatrix} P_{11}^1 & 0 \\ 0 & P_{22}^2 \end{pmatrix} + R \begin{pmatrix} P_{12}^1 & 0 \\ 0 & P_{21}^2 \end{pmatrix} \begin{pmatrix} P_{11}^2 & 1 \\ 1 & P_{22}^1 \end{pmatrix} \begin{pmatrix} P_{21}^1 & 0 \\ 0 & P_{12}^2 \end{pmatrix} \quad . \tag{8}$$

Obviously, by repeated combination of subsequent objects, the final matrix may become quite complicated, but then it represents the whole chain as one single 2-port object; the complexity of the matrix poses no problem in a numerical treatment.

n-port objects

Definition

The above formalism allows the representation of linear (one-dimensional) systems of coupled cavities. An interferometer containing beam splitters requires a generalization: The normal incidence condition for mirrors, as requested above, is not fulfilled in the case of a beam splitter, so the reflected beam from one side and the transmitted beam from the other no longer coincide. Thus one obtains an object with 4 possible inputs/outputs (4-port object).

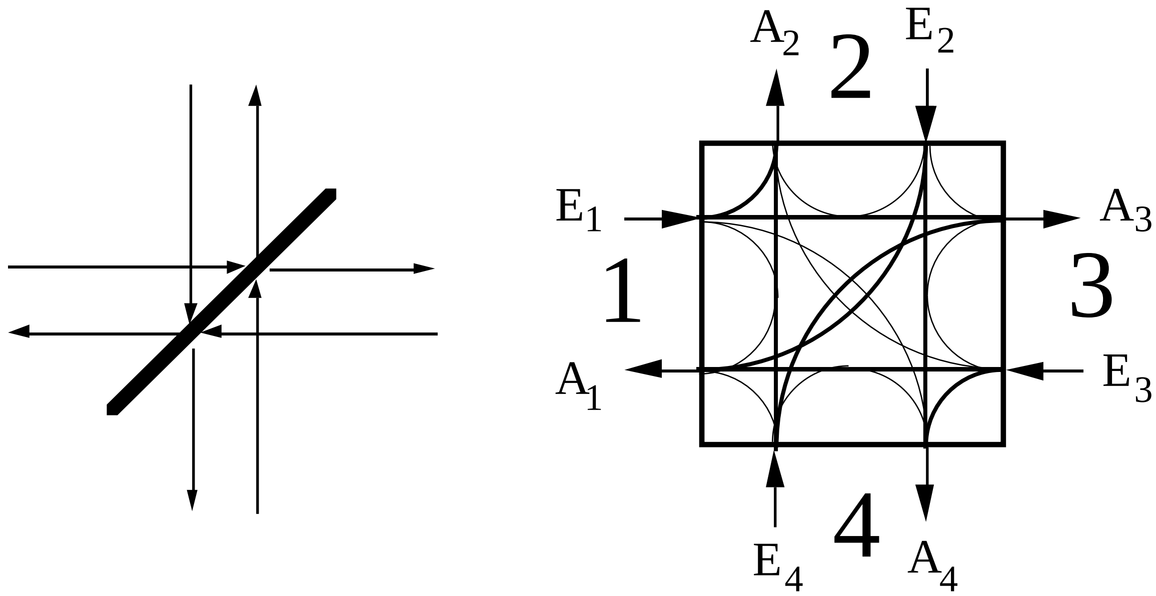

Fig. 3: Representation of a beam splitter as a 4-port object. The paths theoretically possible in a general 4-port object are added as thin lines.

The calculations will be done for the general case of n-port objects (nPs), which aids clarity without adding any difficulty. Obviously, an n-port object can be represented by a nxn matrix **M**, and the input and output field vectors are obtained in the same way as before:

$$\begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} = \begin{pmatrix} M_{11} & \dots & M_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ M_{n1} & \dots & M_{nn} \end{pmatrix} \begin{pmatrix} E_1 \\ \cdot \\ \cdot \\ \cdot \\ E_n \end{pmatrix} \quad , \tag{9}$$

where the element M_{ij} gives the field transmission (reflection) from port j to i. The matrix for a beam splitter (tilted mirror), with ports numbered as in Fig. 3, is then

$$\mathbf{S} = \begin{pmatrix} 0 & r & it & 0 \\ r & 0 & 0 & it \\ it & 0 & 0 & r \\ 0 & it & r & 0 \end{pmatrix} \quad . \tag{10}$$

Concatenation

The number of ports of a combination of two general objects is the sum of the two individual dimensions n and m, minus the two ports by which they are coupled (n+m-2). This most general case, however, is of little practical interest to us, so we will limit ourselves to the case m=2, which allows e.g. the coupling of mirrors and propagators to a beam splitter. The dimension of the combined object is then (n+2-2) = n. One beam splitter coupled to another can be treated in the same way, because, before calculating the composed object, it can be reduced to a 2-port object, as will be explained below.

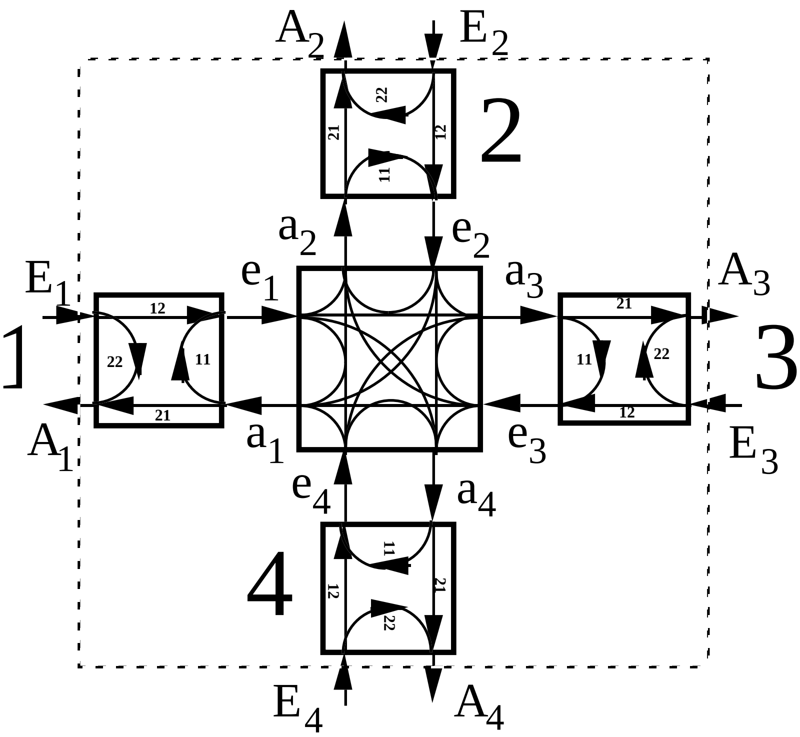

Fig. 4. Combination of an n-port object with 2-port objects (n=4). For symmetry reasons, the port numbering of element 1 is reversed with respect to Fig. 2.

The relations for the input/output vectors are (see Fig. 4 for definitions):

$$\begin{aligned} \mathbf{a} &= \mathbf{M}\,\mathbf{e} \\ \mathbf{A} &= \mathbf{L}\,\mathbf{E} \\ \mathbf{e} &= (\mathbf{P_{12}})\,\mathbf{E} + (\mathbf{P_{11}})\,\mathbf{a} \\ \mathbf{A} &= (\mathbf{P_{22}})\,\mathbf{E} + (\mathbf{P_{21}})\,\mathbf{a} \quad , \end{aligned} \tag{11}$$

where **M** and **L** are the original and the combined 4-port matrix, and $(\mathbf{P_{ij}})$ is defined by

$$(\mathbf{P_{ij}}) = \begin{pmatrix} P^1_{ij} & 0 & \dots & 0 \\ 0 & P^2_{ij} & \dots & 0 \\ . & . & & . \\ 0 & 0 & \dots & P^n_{ij} \end{pmatrix} \; ; \tag{12}$$

$\mathbf{P^k}$ is the 2-port object coupled to port k of **M**. From this follows the combined matrix

$$\mathbf{L} = (\mathbf{P_{22}}) + (\mathbf{P_{21}}) \cdot \left(\mathbf{1} - \mathbf{M} \cdot (\mathbf{P_{11}})\right)^{-1} \cdot \mathbf{M} \cdot (\mathbf{P_{12}}) \quad . \tag{13}$$

Note that the combination of two 2-port objects (Eq. 8) is a special case of Eq. 13, if for **M** a propagator with d=0 is taken, and the port numbering of $\mathbf{P^1}$ is reversed (in order to comply with the convention chosen for the n-port case).

Calculation of the field in the system

For a given system, we need the possibility to calculate the (complex) field "transmission" from the entrance (beam input) to a given point (point of interest).

Basic case

For the simplest case depicted in Fig. 5, the solution is straightforward. For the fields at points 1 and 2, one obtains

$$E_1 = E_0\, P^1_{21} + E_2\, P^1_{22} \quad \text{and} \quad E_2 = E_1\, P^2_{11} \quad , \tag{14}$$

from which the transmission factors follow as

$$E_1/E_0 = R\,P^1_{21} \quad \text{and} \quad E_2/E_0 = R\,P^2_{11}\,P^1_{21} \tag{15}$$

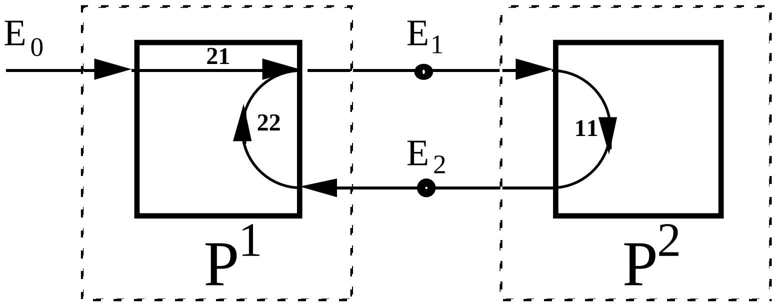

Fig. 5: Basic case for calculating the field in an optical system

General case

The case of a general system, containing one or more n-port elements, can be treated by stepwise reducing it to the basic case. The individual steps to be taken are *(see Fig. 6)*:

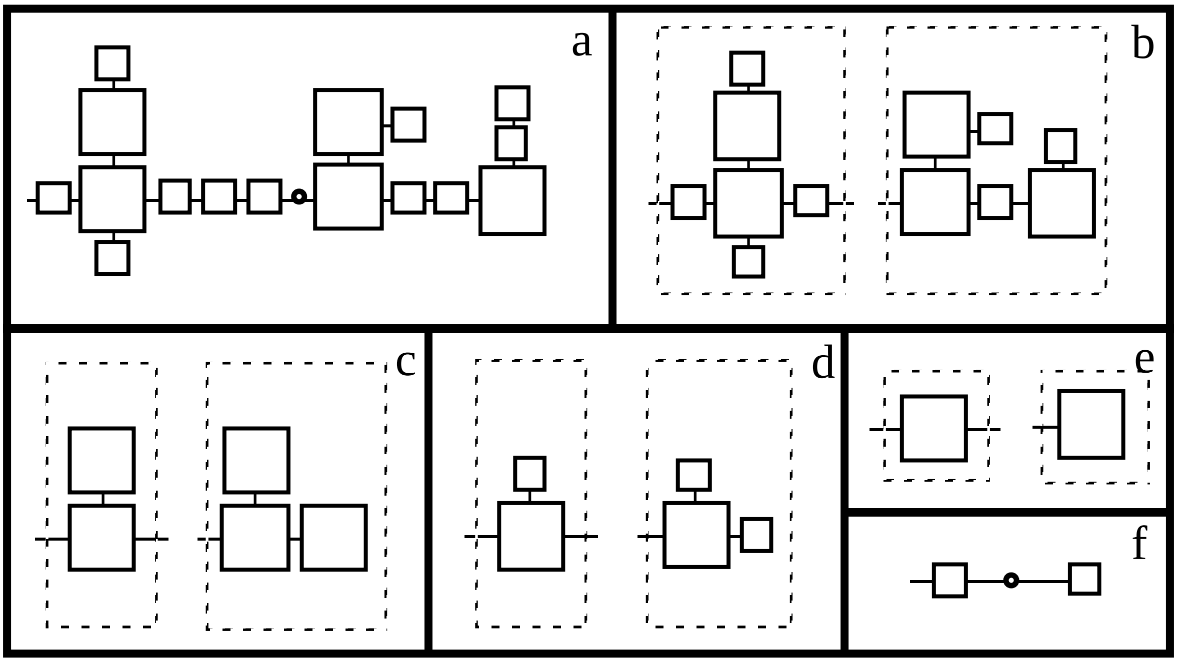

Fig. 6 Example for the reduction of a general system (here with 2-and 4-port objects) to the simple case of Fig. 5.

1. Create two subsystems by splitting the original system at the point of interest *(a→b)*
2. Combine all subchains of 2Ps into a single 2P *(a→b)*. If there are no nPs, proceed to step 6. Otherwise:
3. Combine each nP with its attached 2Ps to one object (Eq. 13) *(b→c)*
4. Reduce the outermost nPs to 2Ps *(c→d)*. This can be done because there is only one field input and one/no interesting output for the first/second subsystem, so at least (n–2) ports of every nP are unused. Thus one obtains the 2P

$$\begin{pmatrix} M_{jj} & M_{jk} \\ M_{kj} & M_{kk} \end{pmatrix},$$

where j,k are the used ports of the object. If only one port is used, the choice of k is arbitrary.
5. Repeat steps 3 and 4 until there remains only one 2P per subsystem *(d→f)*
6. Recombine the two subsystems *(e→f)*

Modulation

One main interest in the investigation of a coupled cavity system is to obtain informations on how to stabilize the distances between the optical elements and thus to assure the proper resonance conditions. In the case of a simple cavity, deviations can be detected using the Pound-Drever technique, which requires phase modulation of the input beam and detecting the modulation frequency in the power of the reflected beam. Similar schemes are feasible for more complicated systems. In principle, modulators can be placed within the system, but frontal modulation (modulation of the input beam [3][4]) presents some advantages, and it can be easily simulated with the present method. Modulating a beam generates sideband fields, whose relative phases depend on the modulation technique used (amplitude or phase modulation). The presence of the modulation frequency in the beam power at a given point is detected in the current of a photodiode; the corresponding error signal is generated by multiplying with the modulation voltage using a mixer.

The calculation of the system must thus be repeated for every frequency (wavelength) at the input (3 for the case of weak phase modulation), yielding the carrier and sideband fields at the point of interest (= place of photodiode or beam sampler). By appropriately summing their products, the beat note is calculated, giving the modulation of the diode current (if any), and thus the observed signal for the given configuration. Next, a mirror is displaced, and the signal is calculated anew. Repeating this for different mirrors and different photodiode positions gives a matrix of the signals at different places for the movement of every mirror; this allows to determine which are the optimum places for the photodiodes, and how their signals must be combined in order to isolate error signals for the control of each individual optical component.

The program CAVITY

Based on the principles described above, a program has been written which allows the treatment of general systems of coupled cavities in the quasi-stationary case (slow mirror motions) with a plane wave approximation. The configuration of the system can be entered interactively and is stored in an ASCII file. For subsequent runs, the configuration can be edited interactively. Allowed are 2-port objects (mirrors, propagators, isolators) and 4-port objects (beam splitters). All ports of a beam splitter may be connected, so there may be e.g. a mirror (dual recycling [5] and resonant sideband extraction [6]) or a cavity (tuned dual recycling [7]) at the dark fringe output of a GW interferometer. Several beam splitters may be used, which in the case of VIRGO permits e.g. adding beam sampling plates. Arbitrary sideband configurations are possible at the input, including AM, PM and SSB modulations. The point of interest can be specified; in the modulated case, the carrier/sideband fields (and phases) themselves can be calculated, or specific superpositions (beat notes), as well as the signal-to-noise ratio. All numerical values of the configuration (object parameters like distance and reflectivity, and input field parameters like frequency, amplitude, modulation index) can be swept by specifying a range instead of a single value. In this case, the output is in form of a graph giving the field (or signal) at the point of interest vs. the swept variable(s), like resonance curve, Pound-Drever locking signal etc. Repeated calculations of variations of a given configuration (e.g. for the calculation of signal matrices) can be automated using a programmed mode.

The program is written in C++ language; on a 486/50 PC it needs 1.5 seconds for the calculation of a 5x5 signal matrix for the VIRGO interferometer with simple frontal modulation (5 lengths are swept; for each length the signal is observed on 5 diodes).

Loops (ring cavities) cannot be treated with the present version.

Modal propagation

The plane wave propagation formalism presented here can be extended to treat non-uniform beam cross sections by approximating the beam by a superposition of a certain number of Gaussian modes. Each element of the matrix describing an optical element would then itself be a matrix of the form

$$\begin{pmatrix} t_{00\text{-}00} & t_{00\text{-}01} & \cdots & t_{00\text{-}kl} \\ t_{01\text{-}00} & t_{01\text{-}01} & \cdots & t_{01\text{-}kl} \\ . & . & & . \\ t_{ij\text{-}00} & t_{ij\text{-}01} & \cdots & t_{ij\text{-}kl} \end{pmatrix} \quad .$$

$t_{ij\text{-}kl}$ describes the coupling between an incident TEM_{kl} mode and a transmitted / reflected TEM_{ij} mode; these factors have been calculated for various cases of misaligned optical elements in [8]. Such a modal propagation formalism would permit e.g. the calculation of error signals for an automatic alignment system according to the Anderson [9] or Ward technique [10][11].

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